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Bifurcations of periodic solutions of delay differential equations ☆

Maoan Han

Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200030, People's Republic of China

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Abstract

In this paper we develop Kaplan–Yorke's method and consider the existence of periodic solutions for some delay differential equations. We especially study Hopf and saddle-node bifurcations of periodic solutions with certain periods for these equations with parameters, and give conditions under which the bifurcations occur. We also give application examples and find that Hopf and saddle-node bifurcations often occur infinitely many times.

Keywords: Delay differential equation; Periodic solution; Bifurcation

1. Introduction

Kaplan and Yorke [9] originated a qualitative method for studying the existence of 4-periodic solutions of the equation

$$\dot{x}(t) = -f(x(t)) \tag{1.1}$$

and 6-periodic solutions of the equation

$$\dot{x}(t) = -[f(x(t-1)) + f(x(t-2))], \tag{1.2}$$

where f is odd with $f(x) = 0$ for $x = 0$. Then Wen [13] gave conditions for the existence of $\frac{4}{4k+1}$-periodic solutions of (1.1), where k is a nonnegative integer. Ge [3]

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E-mail address: mahan@mail.sjtu.edu.cn.

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first found a condition under which the following equation:

$$\dot{x}(t) = -f(x(t-1)) \tag{1.3}$$

has a number of $\frac{4}{4k+3}$–periodic solutions. Chen [1], Gopalsamy et al. [4], Han [7], Kaplan and Yorke [10] and Wang [12] considered a general delay differential equation of the form

$$\dot{x}(t) = -f(x(t), x(t-r)), \quad r > 0, \tag{1.4}$$

and obtained various conditions for the existence of periodic solutions with period $\frac{4}{4k+1}$ or $\frac{4}{4k+3}$. A common requirement for the function f in (1.4) is the following:

$$f(x,y) = f(-x,y), \quad f(x,-y) = -f(x,y), \tag{1.5}$$

which ensures that the orbits of the related planar system

$$\dot{x}(t) = -f(x,y), \quad \dot{y}(t) = f(y,x) \tag{1.6}$$

are symmetric with respect to both x- and y-axis. One can find more results on the existence of periodic solutions to some delay differential equations in Ref. [2,8,11].

In this paper we consider a delay differential equation of more general form, and obtain conditions for the existence of periodic solutions of period $\frac{4r}{4k+1}$ or $\frac{4r}{4k+3}$ without requiring (1.5). What is more, we study Hopf and saddle-node bifurcations of these kinds of periodic solutions for delay differential equations with parameters.

2.2. *Existence of periodic solutions*

Consider a scalar delay differential equation of the form

$$\dot{x}(t) = F(x(t), x(t-r), x(t-2r)), \tag{2.1}$$

where $r > 0$, F is a C^1 function and satisfies

$$F(x,y,-x) = -F(-x,-y,x). \tag{2.2}$$

Obviously, the form of (2.1) is more general than (1.4). We introduce an ordinary differential equation of the form

$$\dot{x} = F(x,y,-x), \quad \dot{y} = F(y,-x,-y). \tag{2.3}$$

The following lemma is evident.

Lemma 2.1. *Let* (2.2) *hold. Then system* (2.3) *is invariant under the change of variables* $(x,y) \to (-y,x)$ *or* $(x,y) \to (y,-x)$. *In other words,* (2.3) *is invariant under a rotation by an angle* $\pm\frac{\pi}{2}$.

Using the above lemma we can prove

Lemma 2.2. *Let* (2.2) *hold. Suppose* (2.3) *has a T-periodic orbit* $L:(x,y)=(x(t),y(t)),\ 0\leqslant t\leqslant T$, *which surrounds the origin. If L is oriented counterclockwise* (*resp., clockwise*) *then the following formula* (2.4) (resp., (2.5)) *holds*:

$$y(t)=x\left(t-\frac{T}{4}\right),\quad x(t)=-x\left(t-\frac{T}{2}\right), \tag{2.4}$$

$$y(t)=-x\left(t-\frac{T}{4}\right),\quad x(t)=-x\left(t-\frac{T}{2}\right), \tag{2.5}$$

Proof. Let

$$z=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix},\quad z_1(t)=Az(t),\quad A=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

By Lemma 2.1 $z_1(t)$ is also periodic solution of (2.3). Let L_1 denote the corresponding orbit. Then it can be obtained by rotating L by an angle $\frac{\pi}{2}$. Thus, these two closed curves must have points in common, and hence $L_1=L$. Thus $z_1(0)\in L$, and there exists a unique $t_1\in[0,T)$ such that $z_1(0)=z(t_1)$. Note that both $z_1(t)$ and $z_1(t+t_1)$ are solutions of (2.3). The uniqueness of solutions implies that they are the same one. Hence

$$Az(t)=z(t+t_1)\quad \text{for all } t. \tag{2.6}$$

Using this equality we obtain

$$\begin{aligned}-z(t)&=A^2z(t)=Az(t+t_1)=z(t+2t_1),\\ z(t)&=A^4z(t)=A^2z(t+2t_1)=z(t+4t_1).\end{aligned} \tag{2.7}$$

From the second equality in (2.7) we have $4t_1=kT$ for an integer $K\geqslant 0$. Then we have $k\neq 0,\ 2$ from the first and $k<4$ by $t_1<T$. Therefore there are only two possible cases: $k=1$ or 3. That is $t_1=\frac{T}{4}$ or $\frac{3T}{4}$. Note that $z(t_1)$ is obtained by rotating $z(0)$ with an angle $\frac{\pi}{2}$. It is clear that if it is oriented counterclockwise (clockwise) then $t_1=\frac{T}{4}(t_1=\frac{3T}{4})$, and equalities in (2.4) ((2.5)) can be obtained by (2.6). The proof is completed. $\square$

Following [7], a periodic solution $x(t)$ of (2.1) satisfying (2.4) or (2.5) is said to be symmetric. Then by Lemma 2.2, we have the following fundamental theorem.

Theorem 2.1. *Let* (2.2) *hold.* (i) *Suppose* (2.3) *has a periodic orbit L with period T and surrounding the origin. If L is oriented counterclockwise and* $T=\frac{4r}{4k+1}$ (*resp., oriented* *clockwise and* $T = \frac{4r}{4k+3}$) *for an integer* $k \geqslant 0$, *then* (2.1) *has a symmetric periodic solution with period* $\frac{4r}{4k+1}$ (resp. $\frac{4r}{4k+3}$). (ii) *Suppose* (2.1) *has a non-zero symmetric periodic solution with period* T. *Then* (2.3) *has a periodic orbit having period* T *and surrounding the origin, and* $T = \frac{4r}{4k+1}$ *or* $T = \frac{4r}{4k+3}$ *for some nonnegative integer* k.

Proof. We consider the case that L is oriented counterclockwise. The other case can be proved in the same way. Let $(x(t), y(t))$ be a representation of L with $T = \frac{4r}{4k+1}$. By Lemma 2.2,

$$y(t) = x\left(t - \frac{T}{4}\right), \quad x(t) = -x\left(t - \frac{T}{2}\right).$$

It follows that

$$y(t) = x\left(t - kT - \frac{T}{4}\right) = x(t - r),$$

$$x(t) = -x\left(t - 2kT - \frac{T}{2}\right) = -x(t - 2r).$$

Hence, $x(t)$ satisfies (2.1) and the first conclusion follows. The proof of the second one is just similar to Theorem 1 in [7]. This finishes the proof. $\square$

From Theorem 2.1 we have immediately

Corollary 2.1. *Let* F *satisfy* (2.2). *Suppose* (2.3) *has a family of periodic orbits* L_h *for* $h \in J \subset R$, *which surround the origin. Set* $\alpha = \inf_J T_h$ $\beta = \sup_J T_h$, *where denotes the period of* L_h. *If the orbits are oriented counterclockwise* (*resp.*, *clockwise*) *and there exists an integer* $k \geqslant 0$ *such that* $\frac{4r}{4k+1} \in (\alpha, \beta)$ (*resp.*, $\frac{4r}{4k+3} \in (\alpha, \beta)$), *then* (2.1) *has a periodic solution with period* $\frac{4r}{4k+1}$(*resp.*, $\frac{4r}{4k+3}$).

We remark that if $F(x, y, z) = -f(x, y)$ is a function with two variables then all existence results on periodic solutions of (2.1) obtained in [1,3,4,7,10,12,13] are contained in Corollary 2.1.

Example 2.1. Consider the delay differential equation

$$\dot{x}(t) = -2x(t-1) + x^3(t) - 3x(t)x^2(t-1). \tag{2.8}$$

The corresponding planar system (2.3) for (2.8) is

$$\dot{x}(t) = -2y - x^3 + 3xy^2 = -H_y,$$

$$\dot{y}(t) = 2x + 3x^2y - y^3 = H_x, \tag{2.9}$$

where $H = x^2 + y^2 + x^3y - xy^3$. Hence, (2.9) has a family of periodic orbits

$$L_h : H(x,y) = h, \quad 0<h<h_0.$$

Since $H(x,0)$ is unbounded, the equation $H(x,y) = h$ defines a nontrivial curve for any $h > 0$. On the other hand, it is easy to see that for $h > \frac{25}{24}$ the line $y = 2x$ does not intersect with the set defined by $H(x,y) = h$. This means that the equation $H(x,y) = h$ no longer defines a closed curve for $h > \frac{25}{24}$. Hence we can choose $h_0 \in (0, \frac{25}{24})$ such that the limit $\lim_{h\to h_0} L_h = L_{h_0}$ exists and it is a heteroclinic cycle of (2.9) consisting of saddles and orbits connecting them. Thus the period T_h of L_h satisfies

$$\alpha = \inf_{0<h<h_0} T(h) \leqslant \lim_{h\to 0} T(h) = \pi, \quad \beta = \sup_{0<h<h_0} T(h) = \lim_{h\to 0} T(h) = \infty.$$

Then by Theorem 2.1, (2.8) has a periodic solution with period 4. However, Eq. (2.8) does not satisfy condition (1.5).

The above example shows that in the case of $F(x,y,z) = -f(x,y)$, condition (2.2) is really weaker than (1.5) even if (2.3) (or (1.6)) is Hamiltonian.

Remark 2.1. Similar to Theorem 2 in [7], we can discuss the uniqueness of symmetric periodic solutions. More precisely, if (2.2) holds and Eq. (2.3) has a family of periodic orbits in an open annulus S not containing the origin, then (2.1) has at most one symmetric periodic solution with period $\frac{4r}{4k+1}$ or $\frac{4r}{4k+3}$ and $(x(t), x(t-r)) \in S$ provided $R(x,y)R_1(x,y) \neq 0$ for all $(x,y) \in S$, where

$$R(x,y) = xF_2(x,y) - yF_1(x,y),$$

$$F_1(x,y) = F(x,y,-x), \quad F_2(x,y) = F(y,-x,-y,),$$

$$R_1(x,y) = x^2\frac{\partial F_2}{\partial x} + xy\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\right) - y^2\frac{\partial F_1}{\partial y} - R(x,y).$$

Now we consider the following equation with a vector parameter:

$$\dot{x}(t) = F(x(t), x(t-r), x(t-2r), a), \tag{2.10}$$

where $a \in I^* \subset R^n$, $n \geqslant 1$. Suppose $F \in C^1$ and satisfies

$$F(x,y,-x,a) = -F(-x,-y,x,a) \tag{2.11}$$

for all $a \in I^*$. Then (2.3) becomes

$$\dot{x} = F(x,y,-x,a), \quad \dot{y} = F(y,-x,-y,a). \tag{2.12}$$

The following two theorems are immediate from Theorem 2.1.

Theorem 2.2. *Let* (2.11) *hold. Suppose there exists a subset* $I \subset I^*$ *such that for each* $a \in I$ (2.12) *has a periodic orbit* $L(a)$ *which surrounds the origin and has the period* $T(a)$. *Define* $V = \{T(a) | a \in I\}$. *If* $L(a)$ *is oriented counterclockwise* (*resp.*, *clockwise*) *for* $a \in I$ *and there exists an integer* $k \geqslant 0$ *such that* $\frac{4r}{4k+1} \in V$ (*resp.*, $\frac{4r}{4k+3} \in V$), *then there exists* $a \in I$ *such that* (2.10) *has a symmetric periodic solution with period* $\frac{4r}{4k+1}$ (*resp.*, $\frac{4r}{4k+3}$).

Theorem 2.3. *Let* (2.11) *holds. Suppose that there exists a subset* I *of* I^*, *an interval* $J(a)$ *with* $a \in I$ *and an integer* $k \geqslant 0$ *such that*

(i) *For each* $a \in I$, (2.12) *has a family of periodic orbits* $L(a, h)$ *for* $h \in J(a)$ *with period* $T(a, h)$ *and having counterclockwise* (*resp.*, *clockwise*) *orientation.*

(ii) *For* $a \in I$ *it holds that*

$$\frac{4r}{4k+1} \in (\alpha(a), \beta(a))) \quad \left(resp., \ \frac{4r}{4k+3} \in (\alpha(a), \beta(a)) \right),$$

where

$$\alpha(a) = \inf_{h \in J(a)} T(a, h), \quad \beta(a) = \sup_{h \in J(a)} T(a, h).$$

Then for all $a \in I$ (2.10) *has a periodic solution* $x_k(t, a)$ *with period* $T_k = \frac{4r}{4k+1}$ (*resp.*, $T_k = \frac{4r}{4k+3}$).

In the rest of this section we give an application to Theorem 2.2. Further results and more applications are given in Section 3.

Example 2.2. Consider

$$\dot{x}(t) = -ax(t-1) + x(t)(1 - x^2(t) - x^2(t-1)), \tag{2.13}$$

where $a \neq 0$. The corresponding system (2.12) becomes now

$$\begin{aligned} \dot{x}(t) &= -ay + x(1 - x^2 - y^2), \\ \dot{y}(t) &= ax + y(1 - x^2 - y^2). \end{aligned} \tag{2.14}$$

Obviously, the circle $x^2 + y^2 = 1$ is an only limit cycle of (2.14) with period $T(a) = \frac{2\pi}{|a|}$. Thus we have $V = (0, \infty)$ for (2.13). Let

$$a_k = \frac{\pi}{2}(4k+1), \quad \bar{a}_k = -\frac{\pi}{2}(4k+3), \ k \geqslant 0.$$

Then by Theorems 2.1 and 2.2 it follows that (2.13) has a unique symmetric periodic solution for $a \in \{a_1, \bar{a}_0, a_1, \bar{a}_1, \dots\}$. The period of the solution is $T_k = \frac{4}{4k+1}$ or $\frac{4}{4k+3}$ if $a = a_k$ or $\bar{a}_k$. In fact, we can find that these solutions are the functions $x_k = \sin(a_k t)$ or $\sin(\bar{a}_k t)$.

3. Hopf and saddle-node bifurcations

Consider Eq. (2.10) again. First, we suppose $F \in C^3$ and

$$F(x, y, -x, a) = A_{10}(a)x + A_{01}(a)y + \sum_{i+j=3} A_{ij}(a)x^i y^j + o(|x, y|^3) \tag{3.1}$$

for (x, y) near (0,0) and $a \in I^* \subset R^n$.

Theorem 3.1. *Let* (2.11) *and* (3.1) *hold. Assume there exists $a_0 \in I^*$ such that*

$$A_{10}(a_0) = 0, \quad A_{01}(a_0) \neq 0, \quad g \equiv A_{12}(a_0) + 3A_{30}(a_0) \neq 0.$$

Then there is a constant $\varepsilon > 0$ and a function

$$T(a) = \frac{2\pi}{|A_{01}(a)|}(1 + O(A_{10}))$$

such that for given integer $k \geqslant 0$ and $0 < |a - a_0| < \varepsilon$ satisfying $gA_{10}(a) < 0$, Eq. (2.10) *has a unique symmetric periodic solution $x_k(t, a) = O(A_{10})$ of period $T(a)$ if* (i) *$A_{01}(a_0) < 0$ and $T(a) = \frac{4r}{4k+1}$, or* (ii) *$A_{01}(a_0) > 0$ and $T(a) = \frac{4r}{4k+3}$.*

Proof. By (3.1), Eq. (2.12) has the form near the origin

$$\begin{aligned} \dot{x} &= A_{10}x + A_{01}y + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3 + \cdots, \\ \dot{y} &= -A_{01}x + A_{10}y - A_{03}x^3 + A_{12}x^2y - A_{21}xy^2 + A_{30}y^3 + \cdots. \end{aligned} \tag{3.2}$$

For $a = a_0$ the origin is a weak focus. By a formula of [5, p. 152] the first focus value of the origin for (3.2) with $a = a_0$ is given by Hopf bifurcation.

$$C_1 = \tfrac{1}{16}[12A_{30}(a_0) + 4A_{12}(a_0)] = \tfrac{1}{4}g \neq 0.$$

The Hopf bifurcation theorem for planar systems implies that (3.2) has a unique limit cycle $L(a)$ near the origin for $|a - a_0|$ small if $gA_{10}(a) < 0$. Note that the linear terms in (3.2) are in normal form and quadratic terms do not exist. It follows that the limit cycle $L(a)$ can be represented as $r = r(\theta, a)$ with $r^2(\theta, a) = O(A_{10})$ in polar coordinates, and hence $T(a) = \frac{2\pi}{|A_{01}|}(1 + O(A_{10}))$. Now the conclusion is clear by Theorem 2.1. The proof is completed. □

Remark 3.1. If $a \in R^2$ then in general the equation $T(a) = \frac{4r}{4k+1}$ or $T(a) = \frac{4r}{4k+3}$ defines a curve passing through a_0. As $gA_{10}(a) < 0$ and $a \to a_0$ on this curve the periodic solution $x_k(t, a)$ in Theorem 3.1 goes to zero. Hence a_0 is a Hopf bifurcation value. As we will see in the following example, there exists an infinitely many number of bifurcation values in general.

Example 3.1. Consider

$$\dot{x}(t) = bx(t-1) + x(t)(a - x^2(t)). \tag{3.3}$$

Take a and b as parameters. Then by Theorem 3.1 it is easy to see that for any given integer $k \geqslant 0$, (3.3) has symmetric periodic solution $x_k(t,a) = O(a)$ if $b = \frac{\pi}{2}(4k+1)(1+O(a))$ and $0 < a \ll 1$ or $b = -\frac{\pi}{2}(4k+3)(1+O(a))$ and $0 < a \ll 1$.

Thus, for (3.3) the set of Hopf bifurcation values on the (a,b) plane is $\{(0, \frac{\pi}{2}(4k+1)), (0, -\frac{\pi}{2}(4k+3)) | k = 0, 1, 2, \dots\}$.

We have seen that at least two parameters are needed in order to make a Hopf bifurcation occur under the conditions of Theorem 3.1. In the following, we assume a is a scalar parameter and give a second set of conditions for Hopf bifurcation. More precisely, we prove

Theorem 3.2. *Let* (2.11) *and* (3.1) *hold with* $a \in I^* \subset R$. *Suppose*

(i) *There exists an interval* $I \subset I^*$ *such that* (2.12) *has a first integral of the form*

$$H(x,y,a) = x^2 + y^2 + O(|x,y|^3) \quad \textit{for } a \in I.$$

(ii) *There exists an endpoint* a_k *of* I *such that*

$$A_{01}(a_k) \neq 0, \quad A_{21}(a_k) + 3A_{03}(a_k) \neq 0, \quad \frac{2\pi}{|A_{01}(a_k)|} = T_k,$$

where $k \geqslant 0$ *is an integer, and*

$$T_k = \begin{cases} \frac{4r}{4k+1} & \textit{if } A_{01}(a_k) < 0, \\ \frac{4r}{4k+3} & \textit{if } A_{01}(a_k) > 0. \end{cases}$$

(iii) *For* $a \in I$, $A_{01}(A_{21} + 3A_{03})(|A_{01}(a_k)| - |A_{01}|) > 0$.

Then Eq. (2.10) *has a Hopf bifurcation at* $a = a_k$ *and has a unique symmetric periodic solution with* (*the least*) *period* T_k *for* $a \in I$ *close to* a_k.

Proof. By condition (i), (2.12) has a family of periodic orbits $L(a,h)$ given by $H(x,y,a) = h$ for $a \in I$ and $0 < h \ll 1$. Let $(x(t,a,h), y(t,a,h))$ be a representation of $L(a,h)$ with period $T(a,h)$. It is obvious that $x(t,a,h), y(t,a,h) = O(\sqrt{h}) \to 0$ as $h \to 0$ uniformly for $a \in I$ near a_k. We claim that

$$T(a,h) = \frac{2\pi}{|A_{01}|}\left(1 - \frac{A_{21} + 3A_{03}}{|4A_{01}|}h + o(h)\right) \quad \text{for } a \in I. \tag{3.4}$$

We use polar coordinates to prove (3.4). Let $x = r\cos\theta$, $y = r\cos\theta$. Then by (3.2) we have

$$\frac{d\theta}{dt} = -A_{01} - \Delta(\cos\theta, \sin\theta)r^2 + o(r^2),$$

where

$$\Delta(x, y) = A_{03}(x^4 + y^4) + 2A_{21}x^2y^2 + (A_{30} - A_{12})(x^3y - xy^3).$$

Note that $L(a, h)$ can be represented as $r = \sqrt{h} + O(h) \equiv r(\theta, h)$ for $0 \leqslant \theta \leqslant 2\pi$. Thus, along the periodic orbit we have

$$\frac{d\theta}{dt} = -A_{01} - \Delta(\cos\theta, \sin\theta)h + o(h).$$

Hence,

$$\begin{aligned} T(a, h) &= \left| \int_0^{2\pi} \frac{d\theta}{A_{01} + \Delta(\cos\theta + \sin\theta)d\theta + o(h)} \right| \\ &= \frac{1}{|A_{01}|}\left[2\pi - \frac{h}{A_{01}} \int_0^{2\pi} \Delta(\cos\theta + \sin\theta)d\theta + o(h) \right]. \end{aligned}$$

It is direct that

$$\Delta(\cos\theta + \sin\theta) = A_{03} + \tfrac{1}{2}(A_{21} - A_{03})\sin^2 2\theta + (A_{30} - A_{12})(\cos^3\theta \sin\theta - \cos\theta \sin^3\theta).$$

Thus,

$$\int_0^{2\pi} \Delta(\cos\theta, \sin\theta)d\theta = \frac{\pi}{2}(A_{21} + 3A_{03}).$$

Then (3.4) follows. Consider the equation $T(a, h) = T_k$. By (3.4) and condition (ii), this equation is equivalent to

$$\frac{A_{21} + 3A_{03}}{4A_{01}} h + o(h) = \frac{1}{|A_{01}(a_k)|}(|A_{01}(a_k)| - |A_{01}|).$$

The implicit function theorem implies that the above equation has a unique solution

$$h = h_k(a) = \frac{4A_{01}(|A_{01}(a_k)| - |A_{01}|)}{|A_{01}(a_k)|(A_{21} + 3A_{01})}(1 + O(|a - a_k|)).$$

Condition (iii) follows that $h_k(a) > 0$ for $a \in I$ near a_k. Therefore we have $T(a, h_k(a)) = T_k$ for $a \in I$ near a_k. Let

$$x_k(t, a) = x(t, a, h_k(a)), \quad y_k(t, a) = y(t, a, h_k(a)).$$

Then the following hold:

(i) $(x_k(t, a), y_k(t, a)) = O(\sqrt{h_k(a)}) = O(\sqrt{|a - a_k|}) \to 0$ as $a \to a_k,\ a \in I$.
(ii) $(x_k(t, a), y_k(t, a))$ is a periodic solution of (2.12) with period T_k for $a \in I$ near a_k.

Hence by Theorem 2.1 $x_k(t, a) = O(\sqrt{|a - a_k|})$ is a unique symmetric periodic solution of (2.10) for $a \in I$ near a_k. The proof is completed. □

Remark 3.2. In practice, the equality $\frac{2\pi}{|A_{10}(a_k)|} = T_k$ is used to find a_k. Condition (iii) is used to determine a_k to be an right- or left-hand side endpoint of I.

Example 3.2. Consider the equation

$$\dot{x}(t) = -ax(t-1)[1 - x^2(t)]. \tag{3.5}$$

It was proved in [6, pp. 260–261] that (3.5) has a Hopf bifurcation at $a = \frac{\pi}{2}$ and has a nonconstant periodic solution for $a > \frac{\pi}{2}$. Using Theorems 3.2 and 2.3 we can get the following conclusions:

(1) Let $B = \{a_k, \bar{a}_k | k = 0, 1, 2, \dots\}$, where $a_k = \frac{\pi}{2}(4k+1)$, $\bar{a}_k = -\frac{\pi}{2}(4k+3)$. Then Eq. (3.5) has a Hopf bifurcation at each point of B.

(2) (i) For each integer $k \geqslant 0$, Eq. (3.5) has a periodic solution $x_k(t,a)$ with period $\frac{4}{4k+1}$ for $a > a_k$; (ii) $x_k(t,a) \to 0$ as $a \to a_k$ from right.

(3) (i) For each integer $k \geqslant 0$, Eq. (3.5) has a periodic solution $\bar{x}_k(t,a)$ with period $\frac{4}{4k+3}$ for $a < \bar{a}_k$; (ii) $\bar{x}_k(t,a) \to 0$ as $a \to \bar{a}_k$ from left.

(4) If $a < \bar{a}_K$ or $a > a_K$ for some $K \geqslant 0$ integer then (3.5) has $K+1$ different nonconstant periodic solutions.

Proof. For (3.5) the system corresponding to (2.12) is

$$\dot{x}(t) = -ay(1-x^2), \quad \dot{y}(t) = ax(1-y^2). \tag{3.6}$$

Let $H(x,y) = x^2 + y^2 - x^2y^2$. This is a first integral of (3.6). It is evident that (3.6) has a center at the origin and has integral lines $x = \pm 1$ and $y = \pm 1$. The equation $H(x,y) = h$ defines a periodic orbit $L(a,h)$ of (3.6) for $0 < h < 1$ and $a \neq 0$. Its period $T(a,h)$ satisfies

$$\lim_{h\to 0} T(a,h) = \frac{2\pi}{|a|}, \quad \lim_{h\to 1} T(a,h) = \infty.$$

Note that $L(a,h)$ is oriented counterclockwise (resp., clockwise) for $a > 0$ (resp., $a < 0$). Also, it is easy to see that

$$\frac{4}{4k+1} \in \left(\frac{2\pi}{|a|}, \infty\right) \left(\frac{4}{4k+3} \in \left(\frac{2\pi}{|a|}, \infty\right)\right) \quad \text{for } k = 0, 1, 2, \dots, K$$

if and only if

$$a > \frac{\pi}{2}(4K+1) = a_K \left(a > -\frac{\pi}{2}(4K+3) = \bar{a}_K\right).$$

Hence, conclusions (2)(i), (3)(i) and (4) follow from Theorem 2.3 directly.

Finally, noting that condition (iii) of Theorem 3.2 becomes $|a| > |a_k|$ now, conclusions (1), (2)(ii) and (3)(iii) follow from Theorem 3.2. $\square$

In the rest of this section we study saddle-node bifurcations of periodic solutions. As before, suppose there exists an open interval $(0, h_0(a))$ such that (2.12) has a first integral $H(x, y, a)$ which gives a family of periodic orbits $L(a, h) : H(x, y, a) = h,\ 0 < h < h_0(a),\ a \in I^*$, with period $T(a, h)$. We will consider two cases: $h_0(a) < \infty$ and $h_0(a) = \infty$. We begin with the case $h_0(a) = \infty$.

Theorem 3.3. *Suppose*

(i) *Eq.* (2.12) *is an analytic system, and* (2.11) *and* (3.1) *hold.*

(ii) *For* (x, y) *near* $(0, 0)$, $H(x, y, a) = x^2 + y^2 + O(|x, y|^3)$.

(iii) *There exists an open interval* $I \subset I^*$ *such that Eq.* (2.12) *has a global center with being periodic* $L(a, h)$ *for all* $h > 0$, $a \in I$, *and* $\lim_{h\to\infty} T(a, h) = 0$, $a \in I$.

(iv) *There exists an endpoint* a_k^* *of* I *with* $k \geqslant 0$ *an integer such that*

$$A_{10}(a_k^*) \neq 0, \quad T_k = \beta(a_k^*) \quad \textit{and} \quad \beta(a) > T_k \ \textit{for}\ a \in I,$$

where

$$\beta(a) = \sup_{h>0} T(a, h), \quad a \in I,$$

and

$$T_k = \begin{cases} \frac{4}{4k+1} & \textit{if} \quad A_{01}(a_k^*) < 0, \\ \frac{4}{4k+3} & \textit{if} \quad A_{01}(a_k^*) > 0. \end{cases}$$

(v) $\frac{2\pi}{|A_{01}(a)|} < T_k$, $A_{10}(a)(A_{21}(a) + 3A_{03}(a)) < 0$ *for* $a \in I$ *or* $a = a_k^*$.

Then Eq. (2.12) *has two different* T_k*-periodic solutions* $x_k^j(t, a)$, $j = 1, 2$ *for* $a \in I$ *and* $x_k^j(t, a) \to x_k^*(t)$ *as* $a \to a_k^*$, $a \in I$, *where* $x_k^*(t)$ *is a periodic solution of saddle-node type for* $a = a_k^*$.

Proof. First, by (3.4) and condition (v) we have

$$T_k > T(a, 0), \quad T(a, h) > T(a, 0) \ \text{for}\ 0 < h \ll 1 \quad \text{and} \quad a \in I \ \text{or}\ a = a_k^*.$$

Also, noting that $\lim_{h\to\infty} T(a, h) = 0$ and by condition (iv) we know that the graph of the function $T = T(a, h)$ on the (h, T) plane is as shown in Fig. 1 for $a \in I$.

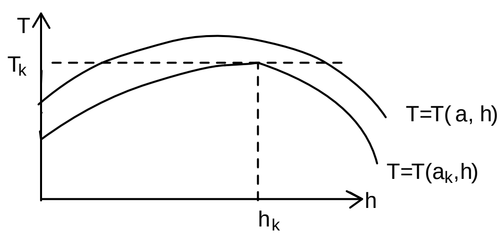

Fig. 1. Graph of $T(a, h)$ for $a \in I$.

Hence, by the definition of $\beta(a)$ there exists $h(a) > 0$ for $a \in I$ such that

$$\beta(a) = T(a, h(a)) \quad \text{for } a \in I.$$

Let $h_k = h(a_k^*)$. Then, by condition (iv),

$$\beta(a_k^*) = T(a_k^*, h_k) = T_k \quad \text{and} \quad \beta(a) > T_k \text{ for } a \in I.$$

Since (2.12) is analytic, the function is also analytic.

Hence,

$$T(a, h) < \beta(a) \quad \text{for } a \in I \text{ and } 0 < |h - h(a) \leqslant 1|. \tag{3.7}$$

It follows that the equation $T(a, h) = T_k$ has at least two solutions $h = h_j(a)$ with $h_1(a) < h(a) < h_2(a)$ for $a \in I$ and $h_j(a) \to h_k$ as $a \to a_k^*$.

If, as before, $(x(t, a, h), y(t, a, h))$ is a representation of $L(a, h)$, then the functions $x_k^{(j)}(t, a) \equiv x(t, a, h_j(a))$ satisfy our requirement. Then the proof is completed.

Remark 3.3. Condition (3.7) ensures that $h_j(a) \to h_k$ as $a \to a_k^*$. Note that by Taylor's theorem

$$T(a, h) = \beta(a) + \tfrac{1}{2} T_h''(a, h(a))(h - h(a))^2 + O(|h - h(a)|^3).$$

It follows that (3.7) holds if

$$T_h''(a_k^*, h_k) < 0.$$

In this case, the analyticity of (2.12) is not necessary.

For the case $h_0(a) < \infty$, we have

Theorem 3.4. *Let conditions* (i) *and* (ii) *of Theorem* 3.3 *hold. Suppose the following conditions are also satisfied*:

(i) *There exists an open interval $I \subset I^*$ such that for $a \in I$ the limit of $L(a, h)$ as $h \to h_0(a)$ is a heteroclinic cycle of* (2.12).

(ii) *There exists an endpoint α_k^* of I with $k \geqslant 0$ an integer such that*

$$A_{01}(a_k^*) \neq 0, \quad T_k = \alpha(a_k^*) \quad \textit{and} \quad \alpha(a) < T_k \quad \textit{for } a \in I,$$

where T_k is the same as in Theorem 3.3 *and $\alpha(a) = \inf_{0 < h < h_0} T(a, h)$, $a \in I$.*

(iii) $\frac{2\pi}{|A_{01}(a)|} > T_k$, $A_{01}(a)(A_{21}(a) + 3A_{03}(a))$ *for $a \in I$ or $a = a_k^*$.*

Then the conclusion of Theorem 3.3 *holds.*

Note that $T(a, h) \to \infty$ as $h \to h_0(a)$ by (i). The proof of Theorem 3.4 is just similar to Theorem 3.3. In this case, the graph of $T(a, h)$ is as shown in Fig. 2 for $a \in I$.

Below we give two examples to show an application of Theorems 3.3 and 3.4.

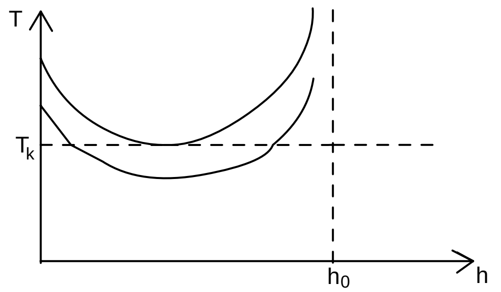

Fig. 2. Graph of $T(a,h)$ for $a \in I$.

Example 3.3. Consider

$$\dot{x}(t) = -ax(t-1)(1 - x^2(t-1) + Ax^4(t-1)), \tag{3.8}$$

where $A > 4$ is a constant. We claim that there exists $T^* > 2\pi$ such that (3.8) has Hopf bifurcation values $a_k = \frac{\pi}{2}(4k+1)$, $\bar{a}_k = -\frac{\pi}{2}(4k+3)$, and saddle-node bifurcation values $a_k^* = \frac{4k+1}{4}T^*$, $\bar{a}_k^* = -\frac{4k+3}{4}T^*$ where $k = 0, 1, \ldots$. Moreover, if $a_k < a < a_k^*$ (resp., $\bar{a}_k^* < a < \bar{a}_k$) then (3.8) has two periodic solutions with period $\frac{4}{4k+1}$ (resp., $\frac{4}{4k+3}$); if $a < a_k$ (resp., $a > \bar{a}_k^*$) then (3.8) has a periodic solution with period $\frac{4}{4k+1}$ (resp., $\frac{4}{4k+3}$).

In fact, the corresponding planar system is

$$\dot{x}(t) = -ay(1 - y^2 + Ay^4), \quad \dot{y}(t) = ax(1 - x^2 + Ax^4) \tag{3.9}$$

which has a first integral $H(x,y) = x^2 + y^2 - \frac{1}{2}(x^4 + y^4) + \frac{A}{3}(x^6 + y^6)$. Since $A > \frac{1}{4}$, the origin is the only singular point of (3.9). Using polar coordinates we have from (3.9)

$$\frac{d\theta}{dt} = a[1 - r^4 S_1(\theta) + Ar^6 S_2(\theta)], \tag{3.10}$$

where $S_1 = \cos^4\theta + \sin^4\theta$, $S_2 = \cos^6\theta + \sin^6\theta$. Hence, the periodic orbit $L(a,h)$ of (3.9) defined by $H(x,y) = h$ can be represented as $r = r(\theta, h)$ and $r(\theta,h) \to \infty$ as $h \to \infty$. Note that $r(\theta,h)$ is independent of a. It follows from (3.10) that $T(a,h) = \frac{T_1(h)}{|a|}$ with

$$T_1(h) = \int_0^{2\pi} \frac{1}{Ar^6(\theta,h)S_2(\theta) - r^4(\theta,h)S_1(\theta) + 1}\, d\theta \to 0 \quad \text{as } h \to \infty.$$

We first consider the case $a > 0$. Then for (3.8) we have

$$A_{01} = -a < 0, \quad A_{01}(A_{21} + 3A_{03}) = -a(0 + 3a) = -3a^2 < 0,$$

and

$$\frac{2\pi}{a}\Big\langle T_h = \frac{4}{4k+1} \Leftrightarrow a \Big\rangle \frac{\pi}{2}(4k+1) = a_k.$$

Let h^* satisfy $T_1(h^*) = \max_{h \geqslant 0} T_1(h) \equiv T^* > 2\pi$. Then noting that $\beta(\alpha) = \frac{T^*}{a}$, we have

$$\beta(\alpha) > T_k \;\Leftrightarrow\; a < \frac{T^*}{T_k} \equiv a_k^*.$$

Thus, it follows from Theorem 3.3 that Eq. (3.8) has two T_k-periodic solutions for $a_k < a < a_k^*$, and a_k^* is a saddle-node bifurcation value. By Theorem 3.2, we know that a_k is a Hopf bifurcation value. Also, for $a < a_k$ we have $\frac{2\pi}{a} > T_k$, $T(a,h) \to 0$ as $h \to \infty$ and hence $T(a,h) = T_k$ has a solution on $a < a_k$. Thus, by Theorem 2.3 (3.8) has a periodic solution for $a < a_k$. Hence, the claim follows for $a > 0$. The case for $a < 0$ is similar.

Example 3.4. Consider

$$\dot{x}(t) = -ax(t-1)[1 - x^2(t-1)]. \tag{3.11}$$

Let $H(x,y) = x^2 + y^2 - \frac{(x^4+y^4)}{2}$. Then (3.11) has two families of periodic orbits which surround the origin. They are

$$L_1(a,h) : H(x,y) = h, \quad |x| < 1, \quad |y| < 1, \;\; 0 < h < \tfrac{1}{2},$$

and

$$L_2(a,h) : H(x,y) = h, \quad x^2 + y^2 > 1, \;\; \tfrac{1}{2} < h < \infty.$$

Denote by $T_i(a,h)$ the period of L_i, and let

$$T(a,h) = \begin{cases} T_1(a,h), & 0 < h < \frac{1}{2}, \\ T_2(a,h), & \frac{1}{2} < h < \infty. \end{cases}$$

Then the graph of $T(a,h)$ for $a \neq 0$ is given in Fig. 3.

Using Theorems 3.4, 3.2 and 2.3 and similar to Example 3.3 we can prove the following conclusion: There exists $T^* < 2\pi$ such that (3.11) has Hopf bifurcation values $a_k = \frac{\pi}{2}(4k+1)$, $\bar{a}_k = -\frac{\pi}{2}(4k+3)$, and saddle-node bifurcation values $a_k^* = \frac{4k+1}{4}T^*$, $\bar{a}_k^* = -\frac{4k+3}{4}T^*$. Moreover, if $a_k^* < a < a_k$ (or $\bar{a}_k < a < a_k^*$) then (3.11) has three periodic solutions; If $a > a_k$ or $a < \bar{a}_k$ then (3.11) has two

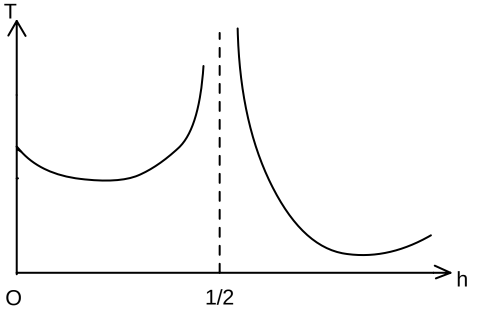

Fig. 3. Graph of $T(a,h)$ for $a \neq 0$.

periodic solutions. Any two of these periodic solutions for all integer $k \geqslant 0$ are different in amplitude.

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